①

$\mathbb{S}^{\sqcap}, \mathbb{S}^{\sqcup}$

$\mathbb{S}^{\sqcap} \circ \mathbb{S}^{\sqcup} = \mathrm{dir}(\ddot{\Theta}^{1/\ell})$

in birat $\mathbb{S}^{\sqcap}_N, \mathbb{S}^{\sqcup}_N$

$\mathbb{S}^{\sqcap\text{-sp}}_N \curvearrowright (\ddot{\Theta}^{1/\ell})^{1/N}$

$\mathcal{I}_N$

$\mathbb{F}^{\Pi}_N \quad \mathcal{D}_{\underline{v}}$

cf. [EtTh, Lem 5.9]

$\ell\,\underline{\Omega} \simeq \Pi_{\underline{v}} / \Pi_{\underline{v}}$

$\to \mathrm{Aut}_{\mathcal{D}}(B_N, \mathcal{D}) \to \mathrm{Aut}_{\mathcal{D}}(B_N) \to \mathrm{Out}(\mathbb{F}_N)$

②

$(K^{\times}) \subset (K^{\times})^{\wedge}$

$\mathcal{O}^{\times}\Gamma(U, \mathcal{O}_U) \to H^1 \to \theta\,\hat{c}$

③ commen. ter.

$N_{\Pi}(I)$ ← decomp.

$C_{\Pi}(I^{\circ})$

rough story

$\mathcal{F}$

$\mathcal{D}$ base

str. of $(\mathcal{D}\text{-})\Theta^{\pm\mathrm{ell}}$NF-Hodge theater

$\boxplus \qquad \boxtimes$

$(\mathcal{D}\text{-})\Theta^{\pm\mathrm{ell}}$-Hodge theater $\qquad (\mathcal{D}\text{-})\Theta$NF-Hodge theater

rough picture

$(\mathcal{D}\text{-})\Theta^{\pm}$-bridge

$(\phi^{\Theta^{\pm}}_{\pm})$

$0 \quad 1 \quad \cdots \quad \ell-1 \quad \gamma^{\Theta^{\pm}}_{\pm}$

$(\mathcal{D}\text{-})\Theta^{\mathrm{ell}}$-bridge

$(\phi^{\Theta^{\mathrm{ell}}}_{\pm})$

$\gamma^{\Theta^{\mathrm{ell}}}_{\pm}$

$X_{\underline{K}}$

$\mathbb{F}_{\ell} \rtimes \{\pm1\}$ $\quad \mathbb{F}^{\times}_{\ell}/\{\pm1\}$

$\boxplus \quad \boxtimes$

$\mathbb{F}^{\rtimes\pm}_{\ell}$-symm. $\qquad \mathbb{F}^{*}_{\ell}$-symm.

$(\mathcal{D}\text{-})\Theta^{\pm\mathrm{ell}}$-HT $\qquad (\mathcal{D}\text{-})\Theta$NF-HT

$(\ldots)\mathcal{HT}^{\Theta}$

$(\mathcal{D}\text{-})\Theta$-bridge

(ϕ^{Θ}_{*})

γ^{Θ}_{*}

$1 \quad 2 \quad j \quad \ell^{*} := \frac{\ell-1}{2}$

$\curvearrowright \mathbb{F}^{*}_{\ell}$

(ϕ^{NF}_{*})

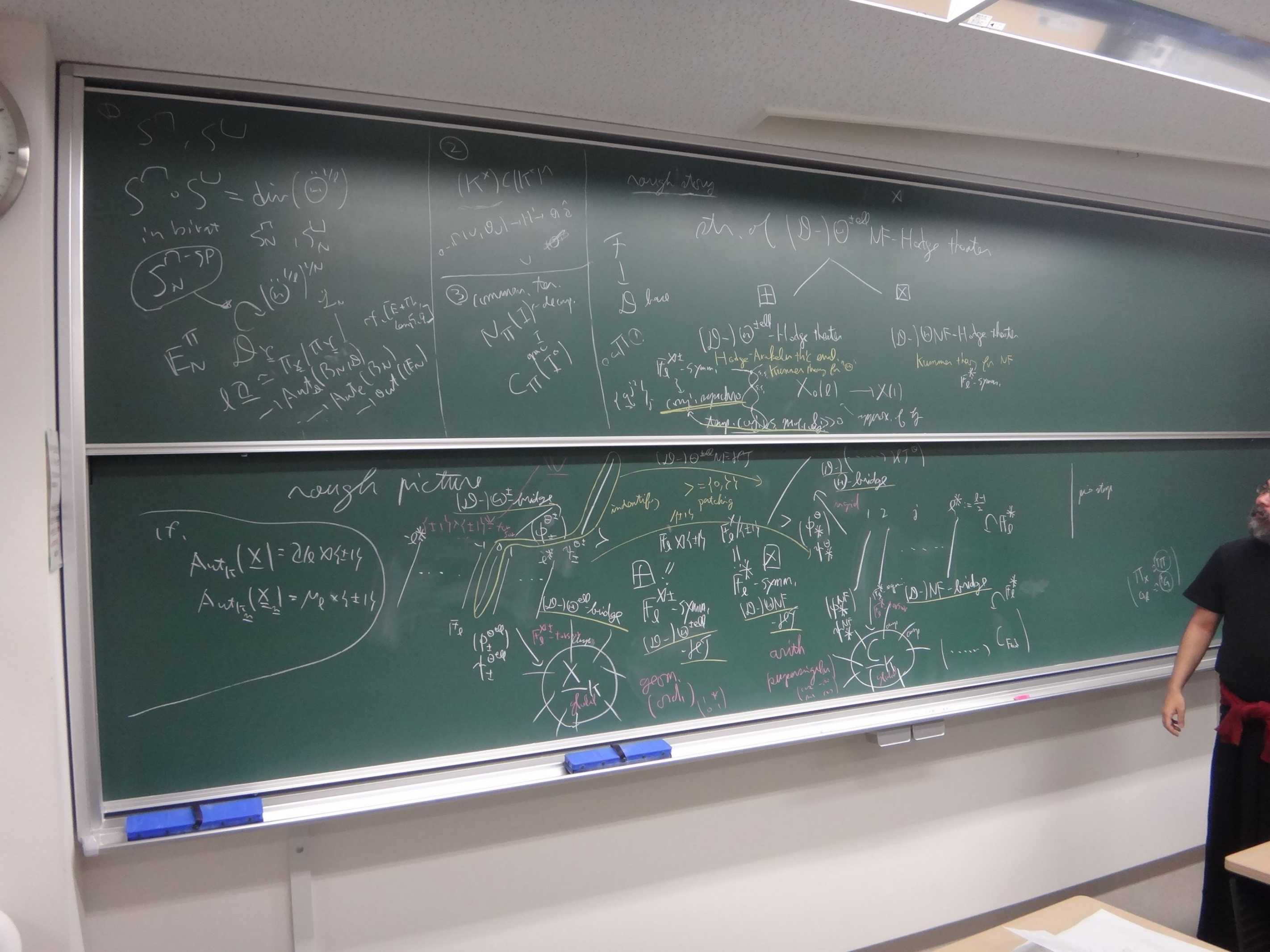
γ^{NF}_{*}

$(\mathcal{D}\text{-})$NF-bridge

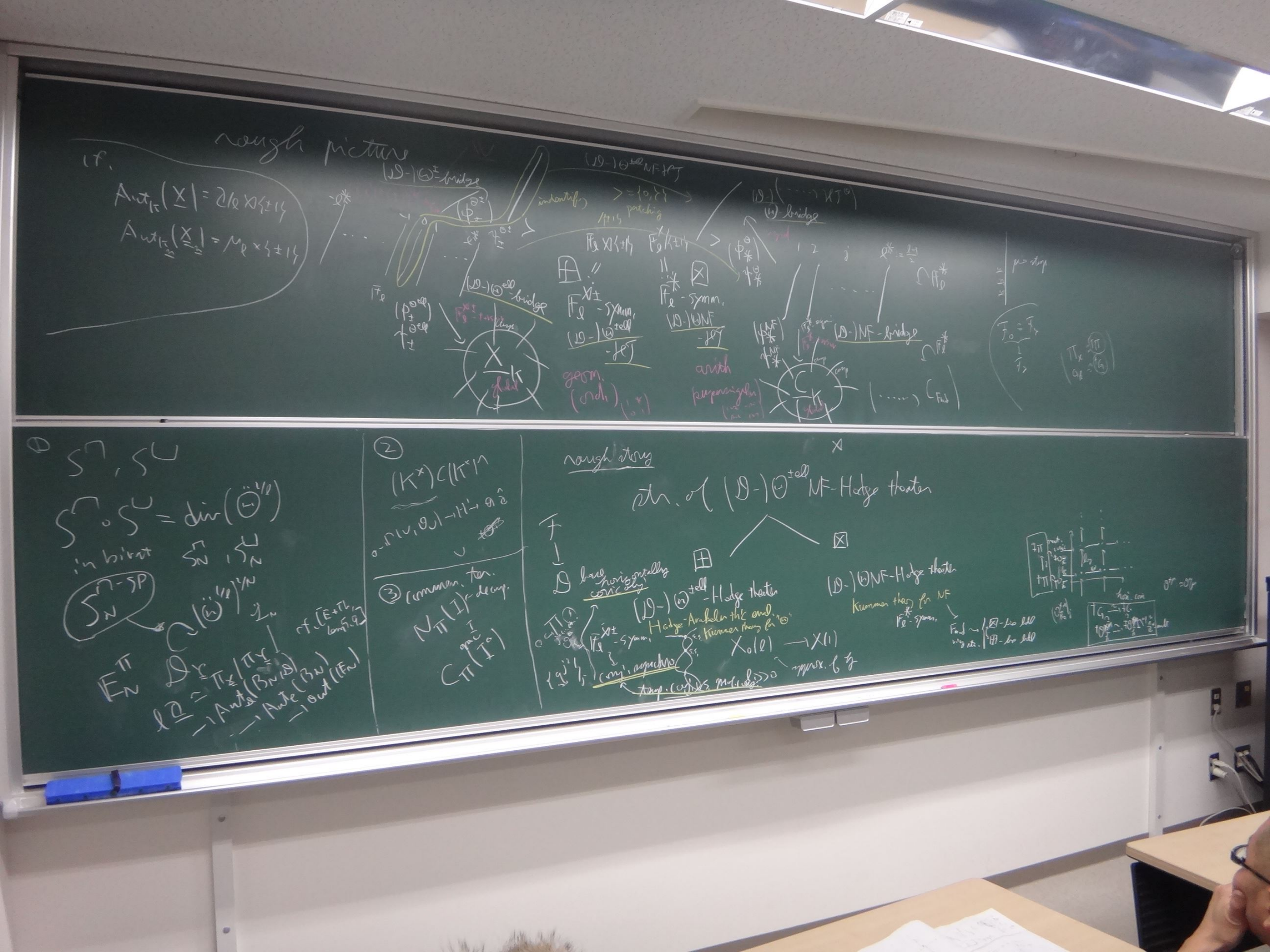
$\curvearrowright \mathbb{F}^{*}_{\ell}$

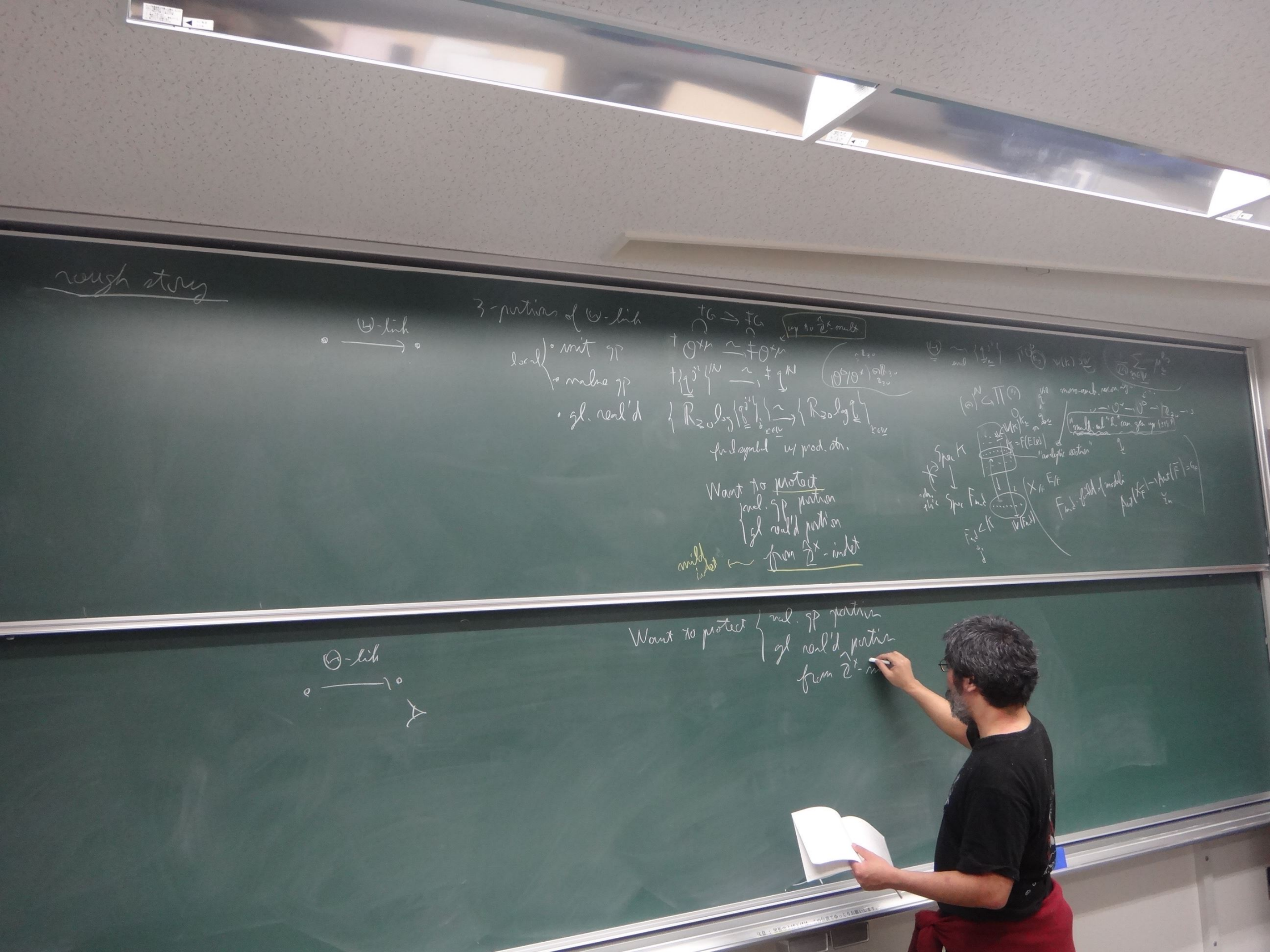
$C_{\underline{K}}$

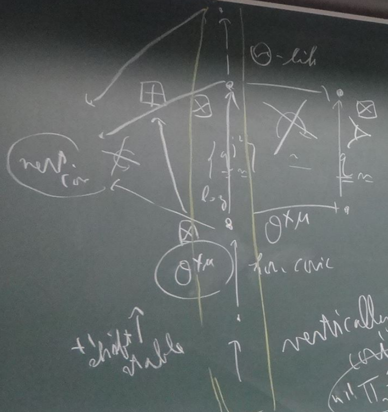
$(\ldots)\ C_{\mathrm{Fmod}}$



in birat
cf.
rough story
Hodge theater
NF-Hodge theater
Kummer theory for NF
rough picture
bridge
identify
patching
NF-bridge
symm.
geom.
arith


rough picture
cf.
(D-)Θ±-bridge
(D-)Θ±ell-bridge
(D-)Θ±ell NF-HT
patching
(D-)Θ-bridge
(D-)NF-bridge
(D-)ΘNF
geom.
arith
in birat
rough story
str. of (D-)Θ±ell NF-Hodge theater
(D-)Θ±ell-Hodge theater
(D-)ΘNF-Hodge theater
Hodge-Arakelov th'c eval.
Kummer theory for NF
approx.


rough story
Θ-link
3-portions of Θ-link
local
unit gp
value gp
gl. real'd
Want to protect
real. gp portion
gl real'd portion
from $\hat{\mathbb{Z}}^\times$-indet
Want to protect
real. gp portion
gl real'd portion
from $\hat{\mathbb{Z}}^\times$-
Θ-link



- $\mathcal{O}^{\times\mu}$ is shared $\boxtimes$

by using
⇝ log-link, we get "$\boxplus$" obj.

⇝ But log-Θ-lattice is highly non-commutative

⇝ use vertical line

⇝ Kummer to vert. coric obj is not compat w/ log-link.

vertically coric

$^{et}\Pi_{\underline{v}} \cong {}^{et}\Pi_{\underline{v}} \cong \cdots$ $\to$ $^{et}\Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\triangleright}(^{et}\Pi)$!

Want to protect $\begin{cases}\text{val. gp portion} \\ \text{gl. real'd portion}\end{cases}$ from $\hat{\mathbb{Z}}^\times$-indet

(unit) log-vol. harmless

⇝ mono-theta env. cycl. rig
⇝ elem. N.T. cycl. rig.

$\mathbb{Q}_{>0} \cap \hat{\mathbb{Z}}^\times = \{1\}$

$\mathcal{O}^\times_{\underline{v}} \subset \frac{1}{p}\log(\mathcal{O}^\times_{\underline{v}})$ common upper bd

$\log(\mathcal{O}^\times_{\underline{v}})$

compatibility by common upper bound
(upper semi-compatibility)

log-Kummer correspondence

totality $\{\text{Kmm} \circ (\log)^n\}_{n \geq 0}$

rough story

$\bullet \xrightarrow{\text{Θ-link}} \bullet$

3-portions of Θ-link

local $\begin{cases}\bullet \text{ unit gp} \\ \bullet \text{ value gp}\end{cases}$

- gl. real'd

$^\dagger G \to {}^\ddagger G$

$^\dagger\mathcal{O}^{\times\mu} \cong {}^\ddagger\mathcal{O}^{\times\mu}$ up to $\hat{\mathbb{Z}}^\times$-mult.

$^\dagger\{\underline{\underline{q}}^{j^2}\}^N \to {}^\ddagger\underline{\underline{q}}^N$

$(\mathcal{O}^\triangleright/\mathcal{O}^\times)^{gp} \otimes \mathbb{R}_{\geq 0}$

$\{\mathbb{R}_{\geq 0}\log(\underline{\underline{q}}^{j^2})\}_{v \in V} \to \{\mathbb{R}_{\geq 0}\log \underline{\underline{q}}\}_{v \in V}$

$\{\mathbb{R}_{\geq 0}\log\}$ fundamental w/ prod. str.

Want to protect $\begin{cases}\text{val. gp portion} \\ \text{gl. real'd portion}\end{cases}$

mild indet ⇝ from $\hat{\mathbb{Z}}^\times$-indet

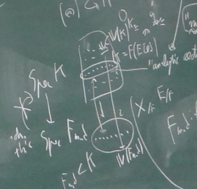
Θ ⇝ env $\{\underline{\underline{q}}^{j^2}\}$

$\frac{1}{|\mathbb{F}_l^*|}\sum_{j} \mu^{j}$

mono-anab. recon.

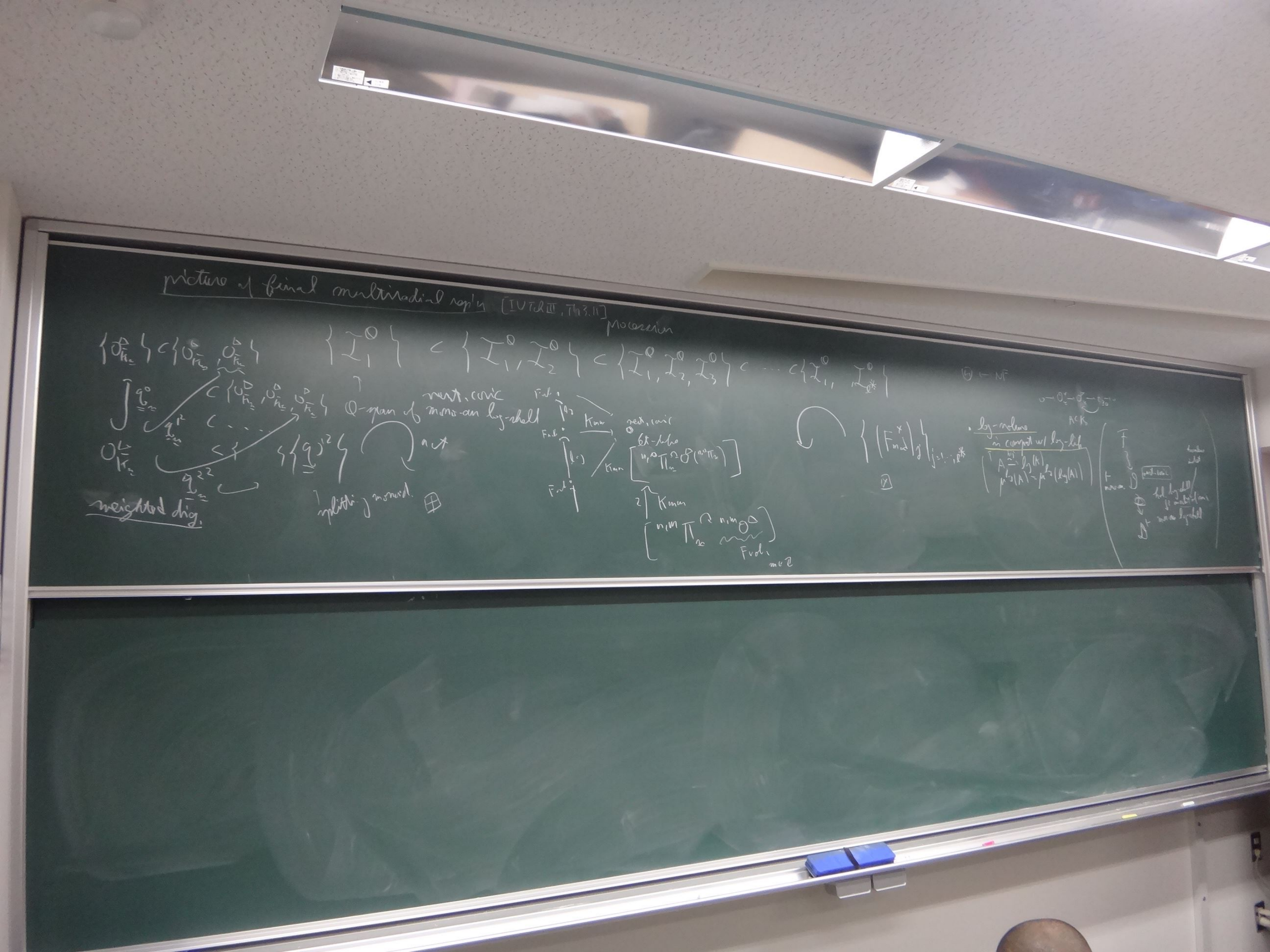
"mult. and "R" can. splitting"

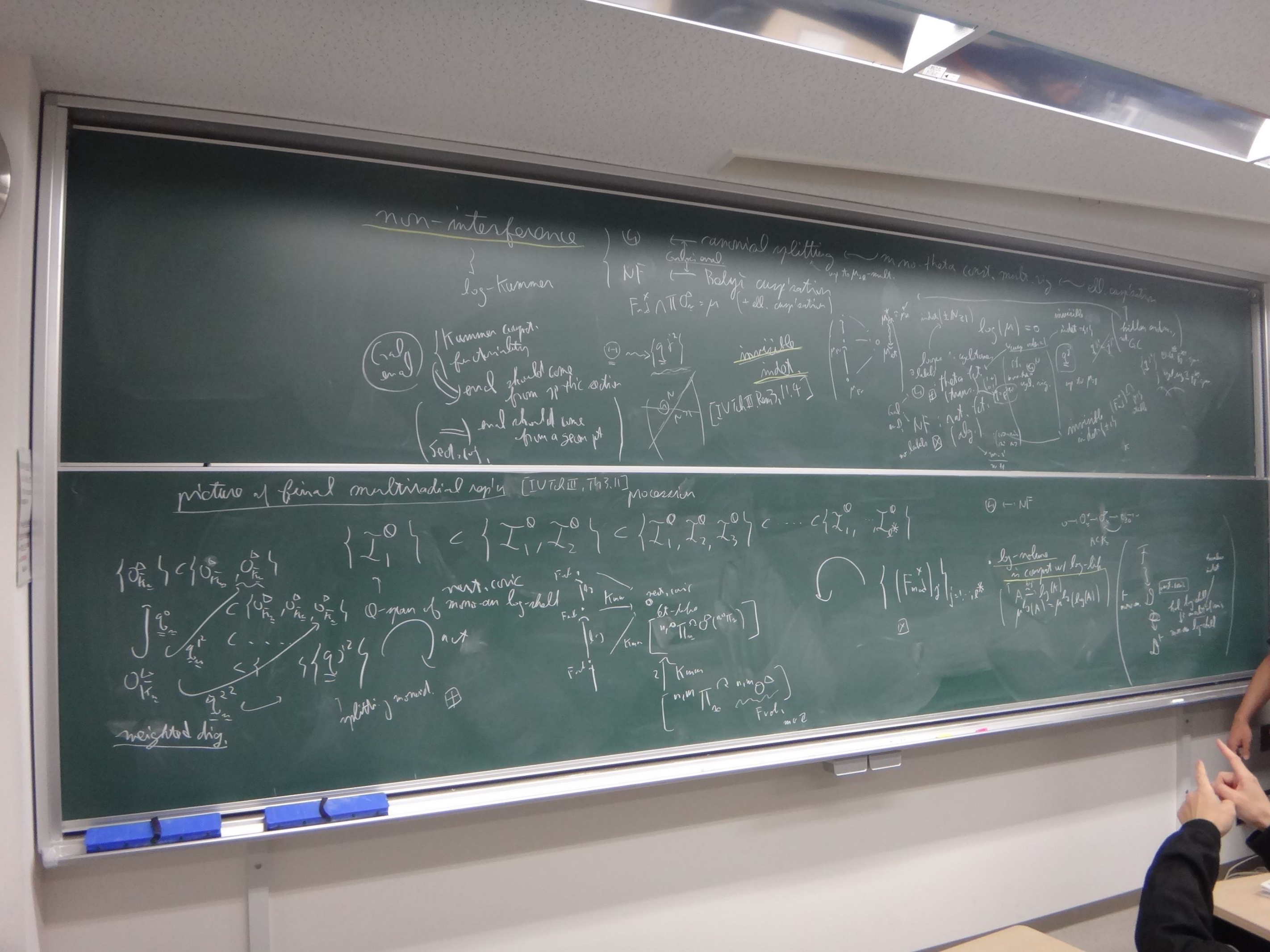
"analytic section"



F_{mod}: field of moduli

$\mathrm{Aut}(X_F) \to \mathrm{Aut}(F)$


picture of final multiradial rep'n [IUTchIII, Thm 3.11]
procession
Q-span of mono-an log-shell
splitting monoid
weighted dig.
Kmm
log-volume
is compat w/ log-link



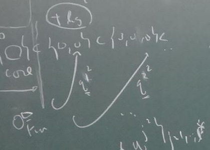
non-interference
log-Kummer
canonical splitting
mono-theta const. mult. rig
ell. cusp'ation
NF
Belyi cusp'ation
$F_{mod}^{\times} \cap \prod \mathcal{O}_v^{\times} = \mu$ (+ ell. cusp'ation)
Kummer compat.
functoriality
end should come from p-adic
end should come from a special pt
invisible
indet.
[IUTchIII, Rem 3.11.4]
$\log(\mu) = 0$
invisible
hidden endom.
picture of final multiradial rep'n [IUTchIII, Th 3.11]
procession
$\{\mathcal{I}_1^{\mathbb{Q}}\} \subset \{\mathcal{I}_1^{\mathbb{Q}}, \mathcal{I}_2^{\mathbb{Q}}\} \subset \{\mathcal{I}_1^{\mathbb{Q}}, \mathcal{I}_2^{\mathbb{Q}}, \mathcal{I}_3^{\mathbb{Q}}\} \subset \cdots \subset \{\mathcal{I}_1^{\mathbb{Q}}, \dots, \mathcal{I}_{l^*}^{\mathbb{Q}}\}$
$\mathbb{Q}$-span of mono-an log-shell
splitting monoid
weighted dig.
Kum
Kmm
$\{(F_{mod}^{\times})_j\}_{j=1,\dots,l^*}$
log-volume
in compat w/ log-link
$\mathcal{O}^{\triangleright} \cdot \mathcal{O}^{\times} \cdot \mathcal{O}^{\times\mu}$
$\mu^{\log}(A) = \mu^{\log}(\log(A))$

cf.
[IUTch I, Fig. 6.5] picture of $\circledast$-tated NF-HT

[IUTch III, Fig. I.6] picture of final multirad rep'n

On cycl. rig.

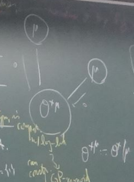
cf. [Pano, Fig 4.1]



	hor. core	cycl. rig.	multi-/uni-radiality	compat. w/ log-link	compat. w/ prof. top.
unit	$\boxplus$ $\mathbb{F}_l^{\rtimes\pm}$-symm.	LCFT	$\mathcal{O}^{\triangleright} \rightsquigarrow$ uniradial (pr. taut. multirad up to $\hat{\mathbb{Z}}^{\times}$)	written semi-compat.	compat.
val. gp	geom. $\rightsquigarrow$ conj. synchro (temp. conj. vs prof. conj.)	mono-theta env.	multirad (use only μ)	compat.	compat. $\rightsquigarrow$ $\lvert\mathbb{F}_l^{\rtimes\pm}\rvert$-compat. is
gl. real'd	$\boxtimes$ $\mathbb{F}_l^{*}$-symm. arith. $\rightsquigarrow$ descend to F_{mod}	$\mathbb{Q}_{>0} \cap \hat{\mathbb{Z}}^{\times} = \{1\}$	multirad. (use only μ)	compat.	incompat. $\mathbb{Q}_{>0} \cap \hat{\mathbb{Z}}^{\times} = \{1\}$

HA-ensal ... -mulna

$\overline{F_{mod}}$



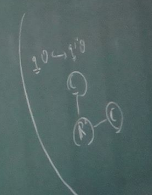
$\mathcal{O}^{\times\mu} := \mathcal{O}^{\times}/\mu$

$\boxed{F_{mod}}$ ⇝ $\boxtimes$-line bdle, $\boxplus$-line bdle

sing. nbhd

cf. [IUTch III, Rem 3.10.1]

log-vol



Kummer detach ((Indet $\longrightarrow$) $\dagger\Theta^{\times\mu} \simeq \ddagger\Theta^{\times\mu}$ up to $\hat{\mathbb{Z}}^\times$

(Indet $\uparrow$) upper semi-compat (common upper bound)

ét trans (Indet $\curvearrowright$) $\dagger G_{\underline{v}} \xrightarrow{\sim} \ddagger G_{\underline{v}}$ + Autom of processions

weak, hol. str., processions

$0 \mid 1 \mid 2 \;\cdots\; l^*$

$\{0\} \subset \{0,1\} \subset \{0,1,2\} \subset \cdots$

indet $= (l^*)!$

indet $= (l^*)^{l^*}$

weak

$\log \mathcal{O}^\times_{\underline{\mu}}$ $\rightsquigarrow$ |Indet| log-diff.

(log-vol.)

$ht \lesssim \boxed{2}(1+\epsilon)(\text{log-diff} + \text{log-cond.})$

$\mathcal{O}_{\mu_2}$

$\{\underline{q}^{j^2}\}^N_{j=1,\dots} \simeq \underline{q}^N$

$\Theta \doteq \underline{q}$ [IUTchI]

Θ-link, HA-...

$\Theta^{\times\mu}$-link, $\Theta^{\times\mu}_{gau}$-link [IUTchII]

log-link

$\Theta^{\times\mu}_{LGP}$-link, $\Theta^{\times\mu}_{lgp}$-link [IUTchIII]

(descent ... ~ 2-dim)

arithmetic

Frob-like ~ ...

ét-like ~ ...

$\{\underline{q}^{j^2}\}_{j=1,\dots,l^*} \simeq \underline{q}$

(1-t)-pilot obj $\xrightarrow{\boxtimes\text{-line bdl}}$ q-pilot obj

log-Kummer

multirad rep'n

ét

permute

0-th column permute

1-st column

$\boxplus$-line bdl

(Indet $\longrightarrow$)
(Indet $\uparrow$)
(Indet $\curvearrowright$)

0-th column

1-st column

one

mono-analytic container $\mathbb{Z}^0$

$\exists$ Indet

possible images

q-pilot obj, log-vol. of Θ-pilot obj

$-|\log \underline{\Theta}| \geq -|\log \underline{q}|$

log-vol. of q-pilot obj

$\cap$

$\mathbb{R}^{\geq 0}$

$-|ht| + |\text{Indet}| \geq 0 \rightsquigarrow |ht| \leq$ (...log-diff + log-cond.)

log-diff + log-cond. [IUTchIV]

$\vdash$: mono-analytic

$\Vdash$: ... mono-analytic

§ 8. NF counterpart of theta evaluation

§ 8.1 Pseudo Monoids

Def 8.1 P : top. space w/ cont. map $P \times P \overset{\text{subset}}{\supset} S \longrightarrow P$

is called <u>top. pseudo monoid</u>

$\overset{\text{def}}{\iff}$ $\exists$ top. abel. gp M ($\leftarrow$ multiplicative)

$\exists$ $\iota : P \hookrightarrow M$ embedding of top. spaces

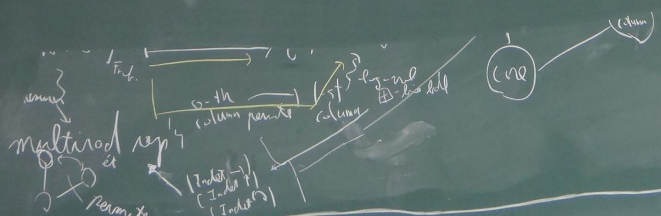
s.t. $\begin{cases} S = \{(a,b) \in P \times P \mid \iota(a)\cdot\iota(b) \in \iota(P) \subseteq M\} \\ \left(M \times M|_S \xrightarrow{\text{gp str.}} \underset{M}{\overset{}{P}}\right) = \text{given } S \to P \end{cases}$

disc. top. $\rightsquigarrow$ <u>pseudo-monoid</u>

P : pseudo-monoid

<u>divisible</u> $\overset{\text{def}}{\iff}$ $\exists M, \exists \iota$: as above s.t.
$\begin{cases} \forall n \geq 1, \forall a \in M, \exists b \in M, b^n = a \\ \forall n \geq 1, \forall a \in M, a \in \iota(P) \iff a^n \in \iota(P) \end{cases}$

<u>cyclotomic</u> $\overset{\text{def}}{\iff}$ $\exists M, \exists \iota$ as above s.t.
$\begin{cases} \cdot\ \mu_M := M_{\text{tor}} (\subset M) \cong \mathbb{Q}/\mathbb{Z} \\ \cdot\ \mu_M \subseteq \iota(P),\ \mu_M \cdot \iota(P) \subseteq \iota(P) \end{cases}$



$-|\log(\Theta)| \leq -|\log \underline{q}|$

$\cap$ $\ell \widetilde{=} (\text{ht})$ log-vol. of q-pilot obj

$\mathbb{R} \cup \{+\infty\}$

$-(\text{ht}) + (\text{Indet}) \gtrsim 0 \rightsquigarrow (\text{ht}) \lesssim (1+\epsilon)(\ldots)$

(log-diff) + (log-cond.)

[IUTch IV]

Def 8.2 ([IUTchI, Rem 3.1.7])

$C_{F_{mod}}$ semi-ell / NF F_{mod} $\quad v$: val of F_{mod}

$\leftarrow F_{mod}$-core

$L := F_{mod}$ or $(F_{mod})_v$, $\quad C_L := C_{F_{mod}} \times_{F_{mod}} L \xrightarrow{\text{fct field}} L_C \subset \overline{L}_C$

$$L^\circ := \begin{cases} F_{mod} & \text{if } L = F_{mod} \text{ or } L = (F_{mod})_v \ v:\text{non-Arch} \\ (F_{mod})_v & \text{if } L = (F_{mod})_v \ v:\text{Arch.} \end{cases}$$

$\overline{L}_C \supset (\,) \supset L_C$

f.g.

proper curve

(1). closed pt is called

a critical point $\overset{\text{def}}{\iff}$ $\in C$ (cpt'fication of $C_{F_{mod}}$)

comes from a 2-tors pt of ell. curve

a strictly critical point $\overset{\text{def}}{\iff}$ $\begin{cases} \text{critical} \\ \text{does not come from cusp} \end{cases}$

(2). $f \in L_C$ (K: Kummer)

K-conic $\overset{\text{def}}{\iff}$
- $f \notin L \Rightarrow f$ has precisely one pole & at least two zeros; each pt
- $(f)_0, (f)_\infty$ def'd / fin. extn of L° & avoids the critical pt
- $f|$ s. crit pt of the cpt'fication of the coarse space of C_L $\in \mu$

(cf. Belyi composition)

Note $f \notin L \Rightarrow$ never both f, f^{-1}: K-conic

$c \in L$, $f \in L_C$, f, cf : K-conic $\Rightarrow c \in \mu$

$\forall$ elt $\in L$ (resp. $\overline{L}$) appears as a value of some K-conic rat. fct. on C_L at some non-critical L-(resp. $\overline{L}$-) valued pt of C_L

(by Hensel's lemma)

(3). $\mathcal{U}_{\overline{L}} := \begin{cases} \overline{L}^\times & \text{if } L = F_{mod} \\ \mathcal{O}_{\overline{L}}^\times & \text{if } L = (F_{mod})_v \end{cases}$

$f \in \overline{L}_C$ $\quad$ ∞K-conic $\overset{\text{def}}{\iff}$ $\exists n \geq 1$, f^n : K-conic

$\infty K\times$-conic $\overset{\text{def}}{\iff}$ $\exists c \in \mathcal{U}_{\overline{L}}$ s.t. cf : ∞K-conic

Note $f \notin L \Rightarrow$ never both f, f^{-1}: k-conic

$c \in L$, $f \in L_c$, f, cf : k-conic $\Rightarrow c \in \mu$

$\forall$ elt $\in L$ (resp. $\bar{L}$) appears as a value of some k-conic rat. fct. on C_L at some non-critical L- (resp. $\bar{L}$-) valued pt of C_L

(by Hensel's lemma)

(3). $\mathcal{U}_{\bar{L}} := \begin{cases} \bar{L}^{\times} & \text{if } L = F_{\text{mod}} \\ \mathcal{O}_{\bar{L}}^{\times} & \text{if } L = (F_{\text{mod}})_v \end{cases}$

$f \in \bar{L}_c$ $\underline{\infty k\text{-conic}} \overset{\text{def}}{\iff} \exists n \geq 1$, f^n : k-conic

$\underline{\infty kx\text{-conic}} \overset{\text{def}}{\iff} \exists c \in \mathcal{U}_{\bar{L}}$ s.t. cf : ∞k-conic

Note · $f \in L_c$: k-conic $\iff$ ∞k-conic

· $f \in \bar{L}_c$: ∞kx-conic is ∞k-conic $\iff$ $f|_{\text{stn. cvd. pt. of pragm. cusp}} \in \mu$

$f \in \mathbb{Q}_{\geq} \cdot L_c$ fin.

{k-conic rat. fct} $\longleftarrow$ pr. monoid

$\cap$

{∞k-conic rat. fct} $\longleftarrow$ disc. ugl. pr. monoids

$\cap$

{∞kx-conic rat. fct}

§8.2 Cycl. Rig via Elem. Number Theory

$K = F(E[\ell])$

$\underline{X}_K \longrightarrow C_K$

as in §7 $C_K := C_F \times_F K$

$C_F := (E_F \setminus \{o\}) / \{\pm 1\}$

F : NF

§ 8.2 Cycl. Rig via Elem. Number Theory (cf. [IUTchI, Ex. 5.1, Def. 5.2])

$K = F(E[l])$

$$\underline{X}_K \longrightarrow C_K$$

$$\text{2:1} \downarrow$$

$$\underline{C}_K$$

as in §7 $\quad C_K := C_F \times_F K$

$C_F := (E_F \setminus \{o\}) / \{\pm 1\}$, $\quad F$: NF

$\mathcal{M}$-approach ([IUTchI, Rem 3.1.2])

$$\Pi_{\underline{C}_K} \supset \Pi_{\underline{X}_K}$$

$\underline{x}$ $\quad \underline{\underline{X}}, \underline{X}$ $\quad \circledcirc$ $\quad \underline{X}$

do not use mono-anab. recon. alg'm in §3 directly to $\Pi_{\underline{C}_K}$

but first recon $\Pi_{\underline{C}_K} \rightsquigarrow \Pi_{\underline{X}_K}$

then, use mono-anab. recon. alg'm in §3 to $\Pi_{\underline{X}_K}$ w/ $\mathrm{Gal}(\underline{X}_K / \underline{C}_K)$-action

$\Pi_{\underline{C}_K} \supset \Delta_{\underline{C}_K}$

$\Delta_{\underline{X}_K} \rightsquigarrow \Pi_{\underline{X}_K}$

(decomp. gp of non-zero labelled cusps)

$\rightarrow \mathrm{Im} \subset \Pi_{\underline{C}_K} / \Delta_{\underline{X}_K}$

$\mathrm{Im} \Leftarrow \Pi_{\underline{X}_K}$

$$1 \to \Delta_\Theta \to \Delta_X^\Theta \to \Delta_X^{ell} \to 1$$

ext'n

$\mathrm{el}^* \in H^2(\Delta_X^{ell}, \Delta_\Theta)$

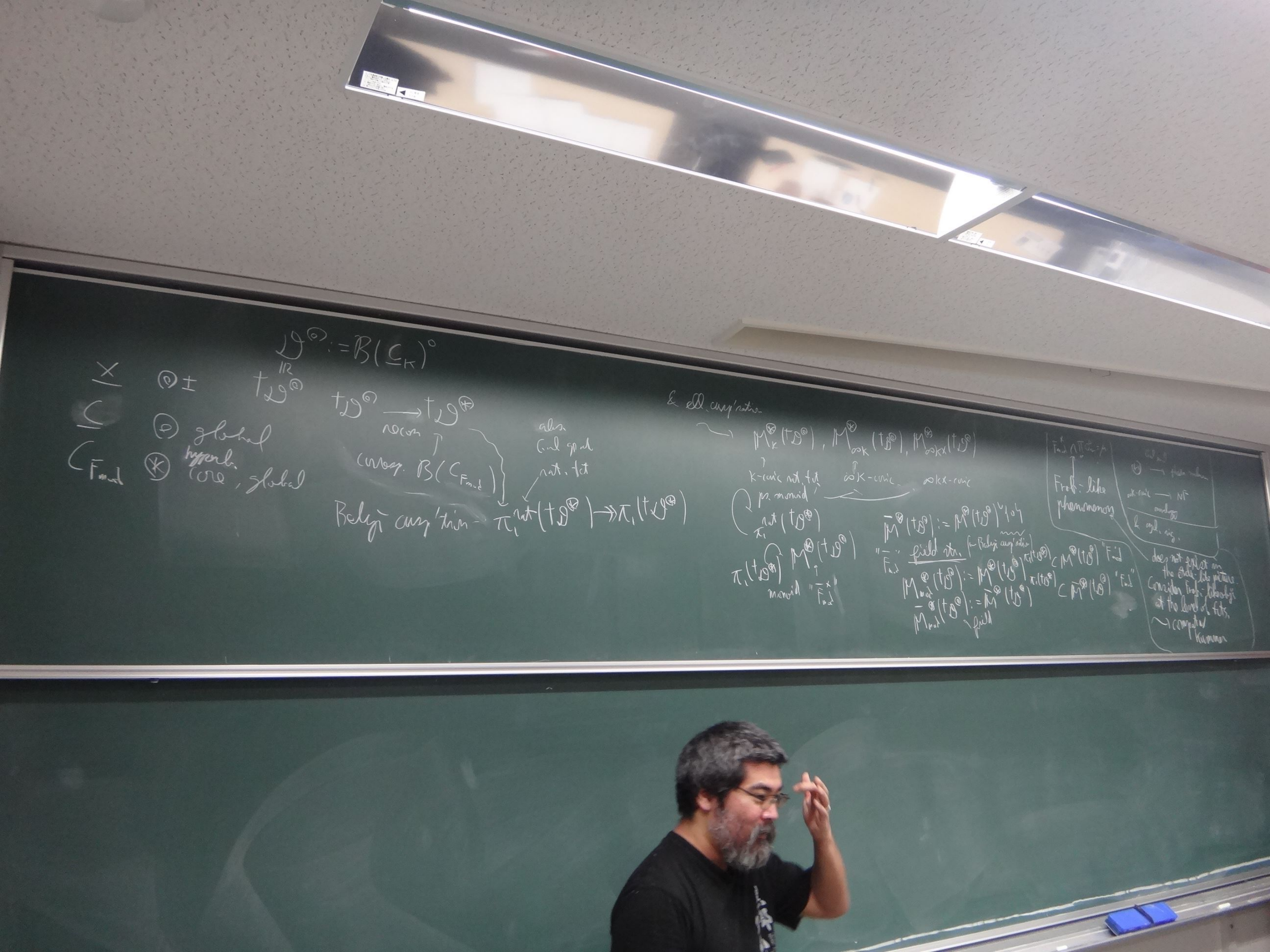
$\cong \mathrm{Hom}(\mu_{\hat{\mathbb{Z}}}(\Pi_X), \Delta_\Theta)$

$\mu_{\hat{\mathbb{Z}}}(\Pi_X) := (H^2(\Delta_X, \hat{\mathbb{Z}}))^*$ $\quad \mu_{\hat{\mathbb{Z}}}(\Pi_X) \xrightarrow{\sim} \Delta_\Theta$

$$\boxed{\mu_{\hat{\mathbb{Z}}}(\Pi_{\underline{X}}) \xrightarrow{\sim} l\Delta_\Theta}$$

we apply mono-anab. alg'm in §3 only to $\underline{X}$

$\mathcal{D}^{\circledcirc} := \mathcal{B}(\underline{X}_K)^{\circ}$


global
core, global
Belyi cusp'tion
& ell. cusp'ation
∞K-conic
∞kx-conic
monoid
field str.
Frob.-like
phenomenon
does not exist in
the étale-like picture.
Consider Frob.-like objects
at the level of fields
Kummer

field str. on $\overline{\mathbb{M}}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$

$\curvearrowright$ gp th'c $\overline{\mathbb{V}}({}^{\dagger}\mathcal{D}^{\circledcirc})$ the set of val'ns ("val. on $\overline{F}_{mod}$")

$\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc}) \rightsquigarrow \underline{\Phi}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$: monoid on ${}^{\dagger}\mathcal{D}^{\circledast}$

$A \mapsto$ stack theoretic arith. line bdl on $\overline{\mathbb{M}}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})^{A}$

$\rightsquigarrow$ model Fr'd $\mathcal{F}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$

(ruby)

given ${}^{\dagger}\mathcal{F}^{\circledast} \cong \mathcal{F}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$

${}^{\dagger}\mathcal{D}^{\circledcirc} \to \mathrm{Base}({}^{\dagger}\mathcal{F}^{\circledast})$

$C_{F_{mod}}$ ${}^{\dagger}\mathcal{D}^{\circledast}$

${}^{\dagger}\mathcal{F}^{\circledcirc} := {}^{\dagger}\mathcal{F}^{\circledast}|_{{}^{\dagger}\mathcal{D}^{\circledcirc}}$

${}^{\dagger}\mathcal{F}^{\circledast}_{mod} := {}^{\dagger}\mathcal{F}^{\circledast}|_{\text{triv. obj.}}$

$A \in \mathrm{Ob}({}^{\dagger}\mathcal{F}^{\circledast}) \rightsquigarrow \mathcal{O}^{\times}(A^{\mathrm{birat}})$

$\mathbb{Z} \geq 0$

$\rightsquigarrow$ run through A : Frob-triv. & Galois obj. $\rightsquigarrow \pi_1({}^{\dagger}\mathcal{D}^{\circledcirc}) \curvearrowright \mathcal{O}^{\circledast \times} = {}^{\dagger}\mathcal{D}^{\circledast}$

$\mathcal{D}^{\circledcirc} := \mathcal{B}(\underline{C}_K)^{\circ}$

$\mathbb{R}$

${}^{\dagger}\mathcal{D}^{\circledcirc}$

$\underline{X}$ $\circledcirc \pm$

$\underline{C}$ $\circledcirc$ global

$C_{F_{mod}}$ $\circledast$ hyperb. core, global

${}^{\dagger}\mathcal{D}^{\circledcirc} \longrightarrow {}^{\dagger}\mathcal{D}^{\circledast}$

nocom

cusp. $\mathcal{B}(C_{F_{mod}})$

also Gal. grp. not. trt

Belyi cusp'tion $\pi_1^{\mathrm{tpd}}({}^{\dagger}\mathcal{D}^{\circledast}) \twoheadrightarrow \pi_1({}^{\dagger}\mathcal{D}^{\circledast})$

& ell. cusp'ation

$\rightsquigarrow \mathbb{M}^{\circledast}_{K}({}^{\dagger}\mathcal{D}^{\circledcirc}),\ \mathbb{M}^{\circledast}_{\infty K}({}^{\dagger}\mathcal{D}^{\circledcirc}),\ \mathbb{M}^{\circledast}_{\infty K \times}({}^{\dagger}\mathcal{D}^{\circledcirc})$

K-conic not. trt $\quad \infty K$-conic $\quad \infty K\times$-conic

pr. monoid

$\pi_1^{\mathrm{tpd}}({}^{\dagger}\mathcal{D}^{\circledast})$

$\pi_1({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$

monoid "$\overline{F}^{\times}_{mod}$"

$\overline{\mathbb{M}}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}) := \mathbb{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})^{\mathrm{gp}} \cup \{0\}$

"$\overline{F}_{mod}$" field str. (← Belyi cusp'ation)

$\mathbb{M}^{\circledast}_{mod}({}^{\dagger}\mathcal{D}^{\circledcirc}) := \mathbb{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})^{\pi_1({}^{\dagger}\mathcal{D}^{\circledast})} \subset \mathbb{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$ "$\overline{F}_{mod}$"

$\overline{\mathbb{M}}^{\circledast}_{mod}({}^{\dagger}\mathcal{D}^{\circledcirc}) := \overline{\mathbb{M}}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})^{\pi_1({}^{\dagger}\mathcal{D}^{\circledast})} \subset \overline{\mathbb{M}}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$ "F_{mod}"

field

$F_{mod}^{\times} \cap \prod \mathcal{O}^{\times}_{\overline{F}_v} = \mu$

Frob.-like phenomenon

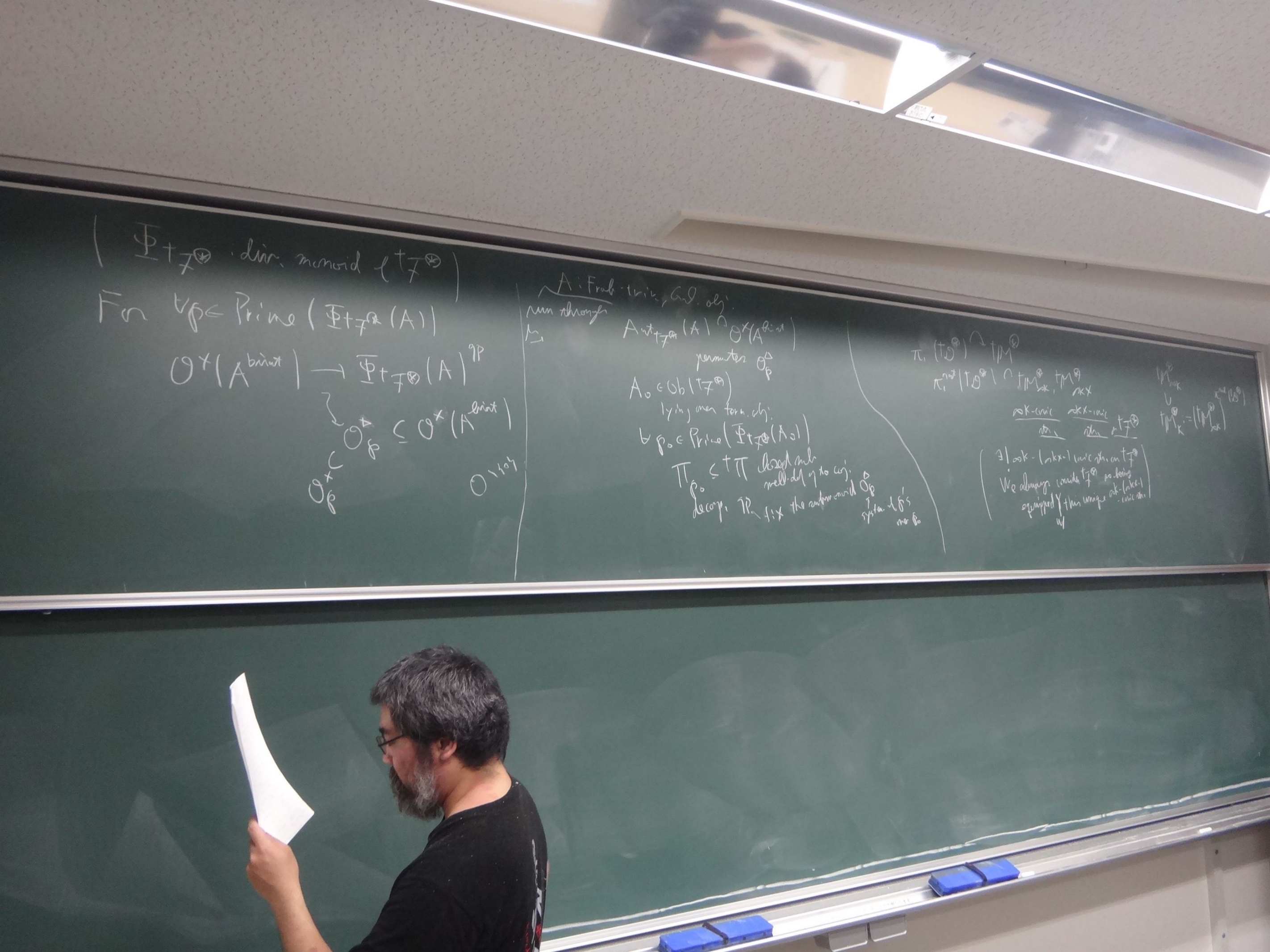
Gal. and

$\circledcirc \to$ étale nature

∞-conic $\to$ NF analogy

& cycl. rig.

does not exist in the étale-like picture. Consider Frob.-like obj. at the level of fields $\rightsquigarrow$ computation Kummer



(Φ_{†F^⊛}: div. monoid of †F^⊛)
For ∀𝔭 ∈ Prime(Φ_{†F^⊛}(A))
O^×(A^{birat}) → Φ_{†F^⊛}(A)^{gp}
O^▷_𝔭 ⊆ O^×(A^{birat})
O^×_𝔭
O^{×▷}
A: Frob-trivial, Gal. obj.
run through
Aut_{†F^⊛}(A) ↷ O^×(A^{birat})
permutes O^▷_𝔭
A_0 ∈ Ob(†F^⊛)
lying over term. obj.
∀𝔭_0 ∈ Prime(Φ_{†F^⊛}(A_0))
Π_{𝔭_0} ⊆ †Π closed sub
well-def up to conj.
decomp. fix the submonoid O^▷_𝔭
system of 𝔭's over 𝔭_0
π_1(†D^⊛) ↷ †M^⊛
π_1^{rat}(†D^⊛) ↷ †M^⊛_{∞κ}, †M^⊛_{∞κ×}
∞κ-coric
∞κ×-coric
str.
str.
†F^⊛
∃! ∞κ-(∞κ×-) coric str. on †F^⊛
We always consider †F^⊛ as being equipped w/ this unique ∞κ-(∞κ×-) coric str.
†M^⊛_{∞κ}
∪
†M^⊛_κ := (†M^⊛_{∞κ})^{π_1^{rat}(†D^⊛)}

$\mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc}) \subseteq \mathbb{M}^{\circledast}_{\infty\kappa\times}({}^{\dagger}\mathcal{D}^{\circledcirc}) \subseteq \varinjlim H^1(H, \mu^{\Theta}_{\hat{\mathbb{Z}}}({}^{\dagger}\mathcal{D}^{\circledcirc}))$

ét-like

Θ-approach

can def $\mu_{\hat{\mathbb{Z}}}(-) = \varprojlim(-)[n]$ for pf-monoid divis. cycl.

$\rightsquigarrow$ obtain (Cycl. Rig, elem. N.T.)

$\mu^{\Theta}_{\hat{\mathbb{Z}}}({}^{\dagger}\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mu_{\hat{\mathbb{Z}}}({}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa}$ (resp. $\infty\kappa\times$))

by imposing condition $\mathcal{O}_{>0} \cap \hat{\mathbb{Z}}^{\times} = \{1\}$

Frob-like

$\mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa}$

(resp. $\infty\kappa\times$) (resp. $\infty\kappa\times$)

$\infty\kappa\times$-coric str.

considering Kummer classes restricting to decomp. gp of str. crit pts

$\infty\kappa$-coric str.

($\mathrm{cf.}$ §3) considering restricting Kummer classes to decomp. gp of non-crit Frd-valued pts

Belyi-cusp'tion

field str. on ${}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}$

Frob-like	coric walls
ét-like	penetrates walls
Kummer	

Similarly, $v \in \mathbb{V}_{\mathrm{mod}}^{\mathrm{non}} := \mathbb{V}(F_{\mathrm{mod}})^{\mathrm{non}}$ case ([IUTchI, Def 5.2 (v)(vi)])

$\underline{v}$: bad

(We also have $v \in \mathbb{V}_{\mathrm{mod}}^{\mathrm{arc}}$ case as well)

(cf.

Π_{ρ_0} decomp.

ρ_0 : inn-ach $\iff$ $cd_p \Pi_{\rho_0} = 3$ for $\exists \infty p$)
$\qquad\qquad\qquad\qquad\qquad\qquad\quad = 1+2$

$$\begin{array}{ccc} \mathbb{V}(K) & \supset & \underline{\mathbb{V}} \\ \downarrow & & \downarrow \wr \\ \mathbb{V}_{\mathrm{mod}} & & \end{array}$$

${}^{\dagger}\mathcal{D}_{\underline{v}} := \mathcal{B}^{\mathrm{temp}}(\underline{\underline{X}}_{\underline{v}})$

${}^{\dagger}\mathcal{D}_{v} = \mathcal{B}^{\mathrm{tp}}(C_v)$

$\pi_1({}^{\dagger}\mathcal{D}_{\underline{v}}) \rightsquigarrow \pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}_{\underline{v}}) \to \pi_1({}^{\dagger}\mathcal{D}_v)$

$\mathbb{M}_v({}^{\dagger}\mathcal{D}_{\underline{v}}), \mathbb{M}_{Kv}({}^{\dagger}\mathcal{D}_{\underline{v}}), \mathbb{M}_{\infty Kv}({}^{\dagger}\mathcal{D}_{\underline{v}}), \mathbb{M}_{\infty K\times v}({}^{\dagger}\mathcal{D}_{\underline{v}})$

"$\mathcal{O}^{\triangleright}_{\bar{K}_v}$" $\curvearrowright \pi_1({}^{\dagger}\mathcal{D}_v)$, $\curvearrowright \pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}_v)$, $\cap \varinjlim_H H^1(H, \mu_{\hat{\mathbb{Z}}}^{\Theta}({}^{\dagger}\mathcal{D}_{\underline{v}}))$

Θ-approach

Frob-like: $\rightsquigarrow {}^{\dagger}\mathbb{M}_{Kv} := ({}^{\dagger}\mathbb{M}_{\infty Kv})^{\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}_v)}$

$\exists !$ ∞K-cuic str. / $\infty K\times$-cuic str. (Cycl. Rig. elem. N.T.)

$\mu_{\hat{\mathbb{Z}}}^{\Theta}({}^{\dagger}\mathcal{D}_{\underline{v}}) \xrightarrow{\sim} \mu_{\hat{\mathbb{Z}}}({}^{\dagger}\mathbb{M}_{\infty K\times v})$

by imposing cond. $\mathbb{M}_{\infty K\times v}({}^{\dagger}\mathcal{D}_{\underline{v}}) \to {}^{\dagger}\mathbb{M}_{\infty K\times v}$

$\infty K\times$-cuic str.

$\}$ étale

∞K-cuic str.

$\}$ Frob-like

field str. on ${}^{\dagger}\mathbb{M}_v^{\circledast}$ ...

$= {}^{\dagger}\mathbb{M}_v^{\circledast}$

$\mathbb{M}_{\infty K}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}) \subseteq \mathbb{M}_{\infty K\times}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}) \subseteq \varinjlim H^1(H, \mu_{\hat{\mathbb{Z}}}^{\Theta}({}^{\dagger}\mathcal{D}^{\circledcirc}))$

Θ-approach

can def $\mu_{\hat{\mathbb{Z}}}(-) = \varprojlim_n (-)[n]$ for μ-monoid, | dim. cycl.

ét-like

$\rightarrow$ obtain (Cycl. Rig. elem. N.T.)

$\mu_{\hat{\mathbb{Z}}}^{\Theta}({}^{\dagger}\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mu_{\hat{\mathbb{Z}}}({}^{\dagger}\mathbb{M}_{\infty K}^{\circledast}$ (resp. $\infty K\times$))

Frob-like

by imposing condition $\mathbb{M}_{\infty K}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} {}^{\dagger}\mathbb{M}_{\infty K}^{\circledast}$ (resp. $\infty K\times$)
(resp. $\infty K\times$)

$\mathcal{O}_{>0} \cap \hat{\mathbb{Z}}^{\times} = \{1\}$

Frob-like	constr. walls
ét-like	penetrate walls
Kummer	

$\infty K\times$-cuic str.

$\}$ considering Kummer classes restricting to decomp. gp of str. crit pts

∞K-cuic str.

(cf. §3) $\}$ consider restricting Kummer classes to decomp. gp of non-crit $\underline{\text{Frob-valued}}$ pts

Belyi-cusp'tion

field str. on ${}^{\dagger}\mathbb{M}_{\mathrm{mod}}^{\circledast}$